

Effective Mass Quantum Systems with Displacement Operator and Coulomb-like Potential

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Abstract

The Schrödinger-like equation written in terms of the displacement operator is solved analytically for a Coulomb-like potential. Starting from the new Hamiltonian, the effects on the bound states and normalized wave functions of the "usual" Coulomb interaction are discussed and given a plot including the variation of first few bound states versus γ .

Keywords: position-dependent mass, translation operator, Coulomb potential, Schrödinger equation

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I. INTRODUCTION

It has been argued that the divergences appearing in field theories can be removed with the idea of noncommutativity and this can be done by using a universal invariant length parameter. This approach has become a central idea in the physical and mathematical points of view [1, 2]. Within the quantum mechanics, the noncommutative coordinates written by the terms of minimum length scale lead to some modifications in position-momentum commutators [3, 4]. This yields an extended Hamiltonian having a position-dependent mass term in kinetic part with some ambiguity parameters α, β, δ satisfying $\alpha + \beta + \delta = 1$ and a Schrödinger equation with effective mass [5-7].

Recently, Filho and co-workers have analyzed a quantum system with position-dependent mass by using a different approach where they suggest a displacement operator given by

$$T_\gamma(dx)|x\rangle = |x + dx + \gamma x dx\rangle, \quad (1)$$

where γ is a real constant describing the mixing between the displacement and the original position state. This operator transforms a well-localized state around x to another well-localized state around $x + (1 + \gamma x)dx$ while all other physical properties remain unchanged [8]. This operator is written explicitly as

$$T_\gamma(dx) = I - \frac{i}{\hbar} \hat{p}_\gamma dx, \quad (2)$$

where \hat{p}_γ corresponds to the generalized linear momentum operator. The commutator between \hat{p}_γ and \hat{x} operator is written as by $[\hat{x}, \hat{p}_\gamma] = (1 + \gamma x)i\hbar$ which gives a generalized uncertainty relation

$$\Delta x \Delta p_\gamma \geq (1 + \gamma \langle x \rangle) \frac{\hbar}{2}, \quad (3)$$

The generalized momentum operator can be given as [8]

$$\hat{p}_\gamma|\alpha\rangle = -i\hbar(1 + \gamma x)\frac{d}{dx}|\alpha\rangle, \quad (4)$$

and the corresponding deformed derivative is written as $D_\gamma = (1 + \gamma x) \frac{d}{dx}$ where $\hat{p}_\gamma = -i\hbar D_\gamma$. In Ref. [9], the generalized momentum operator is written in Hermitian form which enable us to write the Hamiltonian of the system as a Hermitian operator. If one considers the

Hamiltonian operator to be $H = \hat{p}_\gamma^2/2m + V(x)$, one obtains the following Schrödinger-like equation for a single particle

$$\left[-\frac{\hbar^2}{2m} D_\gamma^2 - E + V(x) \right] \phi(x) = 0, \quad (5)$$

or

$$\left[\frac{2}{m(x)} \frac{d^2}{dx^2} + \frac{d}{dx} \left(\frac{1}{m(x)} \right) \frac{d}{dx} + \frac{4}{\hbar^2} [E - V(x)] \right] \phi(x) = 0, \quad (6)$$

with $m(x) = m(1 + \gamma x)^{-2}$. In Refs. [8, 9], the authors have tested their ideas for a free particle and for a particle moving in a one-dimensional infinite well of length L . They have obtained analytical solutions for the above systems. They have discussed the normalization of the wave functions and expectation values of the position. In Ref. [8], they have also studied the dependence of the transmission and tunneling probability on parameter γ for a particle subjected to a potential barrier with height $V_0 > 0$. In Ref. [10], the authors have discussed a general bound state solutions and the corresponding normalized wave functions for a potential function including a quartic and a quadratic term. In the present work, starting from the Schrödinger-like equation given in Eq. (6), we will search the analytical solutions for a particle moving in a Coulomb-like potential of the form A/x . Our aim is to find the bound states and to see the effect of the parameter γ on the energy eigenvalues. We will also find the wave functions with their normalization constants. To do this, we will use the Nikiforov-Uvarov (NU) method [11] which is given in Appendix briefly.

II. ANALYTICAL SOLUTIONS

In order to use the Nikiforov-Uvarov method in the present problem, we change the variable to $z = 1 + \gamma x$ and write the above potential function into Eq. (6) giving

$$\frac{d^2 \phi(z)}{dz^2} + \frac{1-z}{z(1-z)} \frac{d\phi(z)}{dz} + \frac{1}{z^2(1-z)^2} [-a_1^2 z^2 - a_2^2 z - a_3^2] \phi(z) = 0, \quad (7)$$

where

$$-a_1^2 = \frac{2mE}{\gamma^2 \hbar^2}; -a_2^2 = -\frac{4mE}{\gamma^2 \hbar^2} - \frac{2mA}{\gamma \hbar^2}; -a_3^2 = \frac{2mE}{\gamma^2 \hbar^2} + \frac{2mA}{\gamma \hbar^2}, \quad (8)$$

Comparing Eq. (7) with Eq. (A1) in Appendix gives us

$$\tilde{\tau}(z) = 1 - z; \sigma(z) = z - z^2; \tilde{\sigma}(z) = -a_1^2 z^2 - a_2^2 z - a_3^2, \quad (9)$$

Eq. (A6) in Appendix gives the function $\pi(z)$ as

$$\pi(z) = -\frac{z}{2} \pm \sqrt{(1/4 + a_1^2 - k)^2 + (k + a_2^2) + a_3^2}, \quad (10)$$

The constant k is determined by imposing a condition such that the discriminant under the square root has to be zero. This condition gives the constant k as $k = -a_2^2 - 2a_3^2 + a_3$ and the function $\pi(z)$ as $\pi(z) = z(a_3 - 1) - a_3$. The polynomial $\tau(z)$ is calculated from $\pi(z)$ such that its derivative with respect to z must be negative. So, Eq. (A5) in Appendix gives

$$\tau(z) = 1 - 2a_3 + z(2a_3 - 3), \quad (11)$$

with the derivative $\tau'(z) = -3 + 2a_3$. The constant λ is written as $\lambda = -a_2^2 - 2a_3^2 + 2a_3 - 1$ and Eq. (A4) in Appendix gives the eigenvalue as $\lambda_n = n(3 - 2a_3) + n(n - 1)$. By setting $\lambda = \lambda_n$ and using the abbreviations in Eq. (8) we obtain the bound states for a Coulomb-like potential as

$$E_n = -\gamma A - \frac{1}{2n^2} \left[\frac{\gamma \hbar}{2m} n^2 - \frac{A}{\hbar} \right]^2. \quad (12)$$

where is written $n+1 \rightarrow n$. The energy eigenvalues for a Coulomb-like potential are negative and proportional to $E_n \sim 1/2n^2$ for the limit of $\gamma \rightarrow 0$, as expected. The energy value for a bound state is decreases while the parameter γ increases. We give a plot showing the effect of this parameter on the energy spectrum where we use three different values of γ , i.e., $\gamma = 0, 0.01$ and $\gamma = 0.02$ (Fig. 1).

We now focus on the wave functions and their normalization constants. We first compute the weight function from Eqs. (A5) and (A7) in Appendix

$$\rho(z) \sim z^{-a_3}(1 - z), \quad (13)$$

and the first part of the wave functions become

$$\psi_n(z) \sim \frac{1}{z^{-a_3}(1 - z)} \frac{d^n}{dz^n} [z^n(1 - z)^n z^{-a_3}(1 - z)]. \quad (14)$$

The polynomial solutions can be written in terms of the Jacobi polynomials [12]

$$\psi_n(1 - 2z) \sim P_n^{(-2a_3, 1)}(1 - 2z), \quad (15)$$

where we have replaced z by $1 - 2z$. The other part of the wave function is obtained from Eq. (A9) in Appendix

$$\varphi(z) \sim z^{-a_3}(1 - z), \quad (16)$$

Thus, the total wave functions are given in terms of the Jacobi polynomials as

$$\phi(z) = Nz^{-a_3}(1-z)P_n^{(-2a_3,1)}(1-2z). \quad (17)$$

where N is normalization constant. For convenience, we write the Jacobi polynomials in terms of the hypergeometric functions as [13]

$$P_n^{(p,q)}(x) = \binom{n+p}{n} {}_2F_1(-n, n+p+q+1; p+1; \frac{1-x}{2}), \quad (18)$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$. Using the following equality for the hypergeometric functions in the limit of $|z| \rightarrow \infty$ [13]

$${}_2F_1(a, b; c; z) = \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b}, \quad (19)$$

the total wave functions in Eq. (17) is written as

$$\phi(z) = Nz^{-a_3}(1-z) \binom{n-2a_3}{n} [(-1)^n \Gamma_1(z)^n + (-1)^{-n-2+2a_3} \Gamma_2(z)^{-n-2+2a_3}], \quad (20)$$

where

$$\Gamma_1 = \frac{\Gamma(2n+2-2a_3)\Gamma(1-2a_3)}{\Gamma(n+2-2a_3)\Gamma(2n+1-2a_3)}; \Gamma_2 = \frac{\Gamma(2a_3-2-2n)\Gamma(1-2a_3)}{\Gamma(-n)\Gamma(-n-1)}, \quad (21)$$

In order to use the normalization condition $\int_{-\infty}^{+\infty} |\phi(z)|^2 dz = 2 \int_0^{+\infty} |\phi(z)|^2 dz = 1$, we chose a new variable defined as $\xi = z/(1+z)$ ($0 < \xi < 1$) and use the following equality [14]

$$\int_0^1 t^{a'-1} (1-t)^{a''-1} (1-t\eta)^{-a'-a''} dt = B(a', a'') {}_2F_1(a' + a'', a'; a' + a''; \eta), \quad (22)$$

where $B(a', a'')$ is the Beta integral [14] and defined as

$$B(a', a'') = \int_0^1 t^{a'-1} (1-t)^{a''-1} dt, \quad (23)$$

Setting $\eta = 2$ in Eq. (22) and using the normalization condition, we obtain the normalization constants as

$$N = \frac{1}{\sqrt{2}} [(-1)^{2n} \Gamma_1^2 A' + 2(-1)^{-2+2a_3} \Gamma_1 \Gamma_2 A'' + (-1)^{2(-n-2+2a_3)} \Gamma_1^2 A''']^{-1/2}, \quad (24)$$

where

$$\begin{aligned} A' &= B(3-2a_3, -5+2a_3) {}_2F_1(-2, 3-2a_3; -2, 2); A'' = B(-1, -1) {}_2F_1(-2, -1; -2; 2) \\ A''' &= B(-2n-3+22a_3, 2n+1-22a_3) {}_2F_1(-2, -2n-1+22a_3; -2; 2). \end{aligned} \quad (25)$$

III. CONCLUSION

Starting from the Schrödinger-like equation written in terms of the translation operator, we have analyzed the changes of the bound states of a Coulomb-like potential. We have computed the corresponding normalized wave functions analytically. We have also given a plot to see the variation of the bound states according to the parameter γ . We have found that our results obtained for the bound states are in agreement with the ones obtained for the "usual" Coulomb potential as $\gamma \rightarrow 0$.

IV. ACKNOWLEDGMENTS

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Appendix A: The Summary of the Nikiforov-Uvarov Method

The Schrödinger equation can be transformed by using appropriate coordinate transformation into following form

$$\frac{d^2\phi(y)}{dy^2} + \frac{\tilde{\tau}(y)}{\sigma(y)} \frac{d\phi(y)}{dy} + \frac{\tilde{\sigma}(y)}{\sigma^2(y)} \phi(y) = 0, \quad (\text{A1})$$

where $\sigma(y)$ and $\tilde{\sigma}(y)$ are polynomials with second-degree, at most, and $\tilde{\tau}(y)$ is a polynomial with first-degree [11]. We define a transformation for the total wave function as

$$\phi(y) = \psi(y)\varphi(y), \quad (\text{A2})$$

Thus Eq. (A1) is reduced to a hypergeometric type equation [11]

$$\sigma(y) \frac{d^2\psi(y)}{dy^2} + \tau(y) \frac{d\psi(y)}{dy} + \lambda\psi(y) = 0. \quad (\text{A3})$$

We also define the new eigenvalue for the Eq. (A1) with $\lambda = k + \pi'(z)$ as

$$\lambda = \lambda_n = -n\tau'(y) - \frac{n(n-1)}{2} \sigma''(y), \quad (n = 0, 1, 2, \dots) \quad (\text{A4})$$

where

$$\tau(y) = \tilde{\tau}(y) + 2\pi(y), \quad (\text{A5})$$

with

$$\pi(y) = \frac{\sigma'(y) - \tilde{\tau}(y)}{2} \pm \sqrt{(\frac{\sigma'(y) - \tilde{\tau}(y)}{2})^2 - \tilde{\sigma}(y) + k\sigma(y)}, \quad (\text{A6})$$

The derivative of $\tau(y)$ must be negative. $\lambda(\lambda_n)$ is obtained from a particular solution of the polynomial $\psi_n(y)$ with the degree of n . $\psi_n(y)$ is the hypergeometric type function whose solutions are given by [11]

$$\psi_n(y) \sim \frac{1}{\rho(y)} \frac{d^n}{dy^n} [\sigma^n(y) \rho(y)], \quad (\text{A7})$$

where the weight function $\rho(y)$ satisfies the equation

$$\frac{d}{dy} [\sigma(y) \rho(y)] = \tau(y) \rho(y), \quad (\text{A8})$$

On the other hand, the function $\varphi(y)$ satisfies the relation

$$\varphi'(y)/\varphi(y) = \pi(y)/\sigma(y). \quad (\text{A9})$$

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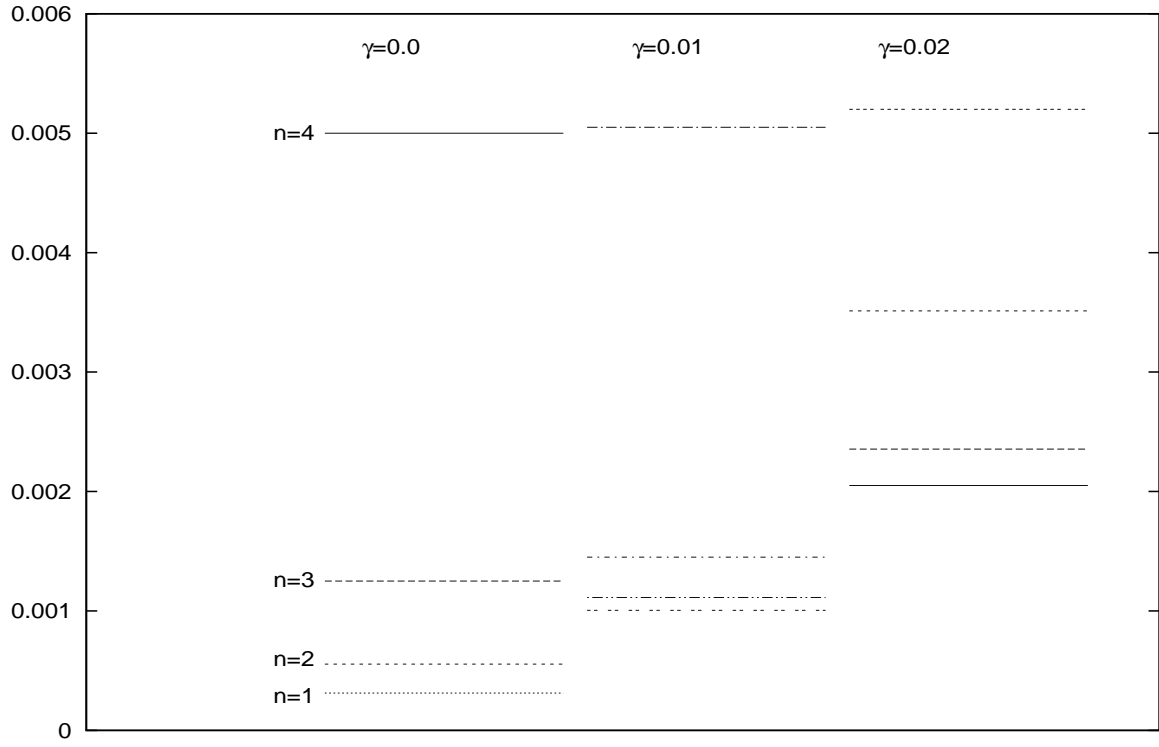


FIG. 1: Energy eigenvalues $|E_n|$ for a Coulomb-like potential versus the parameter γ ($2m = \hbar = 1$).